Problem Set 5

Due: Week 13

1. Suppose the PDF of a random variable X is

$$f(x) = \begin{cases} \frac{1}{8}x & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

   What is the value of t such that $P(X \le t) = 1/4$?

   (a) 1
   (b) 2
   (c) 4
   (d) None of the above

2. Suppose the PDF of a random variable X is

$$f(x) = \begin{cases} ce^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   What is the value of the constant c?

   (a) 1
   (b) 2
   (c) 4
   (d) None of the above

3. Suppose a random variable X has a continuous distribution with the PDF as follows

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   What is the expectation of $1/X$?

   (a) 1
   (b) 2
   (c) 1/2

(d) None of the above

4. Given $X \sim U(2,6)$, what are the mean μ and variance σ^2 of X?

   (a) $\mu = 4, \sigma^2 = \frac{4}{12}$
   (b) $\mu = 4, \sigma^2 = \frac{4}{3}$
   (c) $\mu = 4, \sigma^2 = \frac{4}{12}$
   (d) $\mu = 4, \sigma^2 = \frac{4}{9}$

5. If $X \sim N(\mu, \sigma^2)$, which of the following statements is true?

   (a) $P(X = \mu)$ is maximized
   (b) X is symmetric about μ
   (c) The meanμ is always 0
   (d) The variance σ^2 is always 1

6. For a standard normal distribution $Z \sim N(0,1)$, what is $P(-1 \leq Z \leq 1)$?

   (a) 0.68
   (b) 0.95
   (c) 0.50
   (d) 0.99

7. For a standard normal distribution $Z \sim N(0,1)$, what is the value of z such that $P(Z \leq z) = 0.975$?

   (a) 1.28
   (b) 1.65
   (c) 1.96
   (d) 2.33

8. A bank records the time customers spend waiting in line for a teller. It is known that the waiting time X (in minutes) has a mean of 3 minutes. Assume the waiting time for each customer is independent. A customer complains that they had to wait for more than 10 minutes. Which of the following best describes the probability that a customer will have to wait more than 10 minutes?

   (a) $P(X > 10)$, where $X \sim \text{Unif}(0, 10)$
   (b) $P(X > 10)$, where $X \sim \text{Exp}(1/3)$
   (c) $P(X > 10)$, where $X \sim N(3, 1)$
   (d) $P(X > 10)$, where $X \sim \text{Pois}(3)$

9. Suppose that X, Y, Z are $i.i.d$ random variables and each has the standard normal distribution. Find the value of $P(3X + 2Y < 6Z - 7)$.

10. Suppose that a random sample of 16 observations is drawn from the normal distribution with mean μ and standard deviation 12, and that independently another random sample of 25 observations is drawn from the normal distribution with the same mean μ and standard deviation 20. Let $\bar{X}$ and $\bar{Y}$ denote the sample means of the two sample. Find the value of $P(|\bar{X} - \bar{Y}| < 5)$.

Solutions

1. B. $\int_0^t \frac{1}{8}x dx = \frac{1}{16}t^2 = \frac{1}{4}$.
2. B. This is the exponential PDF, $c = 2$.
3. B. By LOTUS, $E(1/X) = \int_0^1 \frac{1}{x}2x dx = 2$.
4. B
5. B
6. A
7. C
8. B
9. The summation of i.i.d normal RVs are normal. Thus,

$$W = 3X + 2Y - 6Z \sim N(0, 3^2 + 2^2 + 6^2 = 49)$$

The probability in question is equivalent to

$$P(W < -7) = P\left(\frac{W}{7} < \frac{-7}{7}\right) = \Phi(-1) = 0.16$$

10. $\bar{X} = \frac{1}{16}\sum X_i$ where $X_i \sim N(\mu, 12^2)$. Thus,

$$\bar{X} \sim \frac{1}{16}N(16\mu, 16 \times 12^2) = N(\mu, 9)$$

Similarly, $\bar{Y} = \frac{1}{25}\sum Y_i$ where $Y_i \sim N(\mu, 20^2)$. Thus,

$$\bar{Y} \sim \frac{1}{25}N(25\mu, 25 \times 20^2) = N(\mu, 16)$$

Let $W = \bar{X} - \bar{Y} \sim N(0, 9 + 16)$. Therefore,

$$P(|W| < 5) = P(-5 < W < 5) = P(-1 < Z < 1) = 0.68$$